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General Theory of Composite Orthotropic Plates

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In the applied theory of orthotropic plates there are many research works solving the questions of a more general theory of orthotropic plates. Since by Huber the works (1-16) deal also with this problem.

This problem has also been studied at the Institute for Theoretical and Applied Mechanics of the Czechoslovak Academy of Sciences; a brief outline of a general theory formulated by the authors for the calculation of the considered type of composite orthotropic plates is presented in the following sections.

1/General theory of composite orthotropic plates.

Deformation properties of composite orthotropic plates with wall effect are fully decribed by 18 physical and geometric parameters,

$$
\begin{array}{l}
f_{11},\ f_{22},\ S_{11},\ S_{22},\ i_{11},\ i_{22} \\
s_{11},\ s_{22},\ \mathcal{H}_{12},\ \mathcal{H}_{21},\ s_{12},\ s_{21} \\
i_{11},\ i_{22},\ f_{12},\ f_{21},\ i_{12},\ i_{21}
\end{array}
\qquad /1.1/
$$

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Parameters

$$f_{11},\ f_{22},\ s_{11},\ s_{22},\ i_{11},\ i_{22} \qquad /1.2/$$

represent rigidities for unit normal deformations while parameters

$$f_{12},\ f_{21},\ s_{12},\ s_{21},\ i_{12},\ i_{21} \qquad /1.3/$$

represent rigidities for unit shear deformations.
Parameters

$$\nu_{12},\ \nu_{21},\ \varkappa_{12},\ \varkappa_{21},\ \varphi_{12},\ \varphi_{21} \qquad /1.4/$$

are unit values of transverse rigidities and interrelated with parameters /1.2/ by Maxwell's reciprocal theorem

$$f_{11}\nu_{21} = f_{22}\nu_{12}, \qquad s_{11}\varkappa_{21} = s_{22}\varkappa_{12}, \qquad i_{11}\varphi_{21} = i_{22}\varphi_{12} \qquad /1.5/$$

The forces and moments are then given by expressions

$$
\begin{aligned}
N_{xx} &= f_{11}\left(\varepsilon_{xxo} + \nu_{21}\varepsilon_{yyo}\right) - s_{11}\left(\rho_{xxo} + \varkappa_{21}\rho_{yyo}\right),\\
N_{yy} &= f_{22}\left(\varepsilon_{yyo} + \nu_{12}\varepsilon_{xxo}\right) - s_{22}\left(\rho_{yyo} + \varkappa_{12}\rho_{xxo}\right),\\
N_{xy} &= N_{yx} = \frac{f_{12}+f_{21}}{2}\gamma_{xyo} - \left(s_{12}+s_{21}\right)\rho_{xyo}, \qquad /1.6/\\
M_{xx} &= s_{11}\left(\varepsilon_{xxo} + \varkappa_{21}\varepsilon_{yyo}\right) - i_{11}\left(\rho_{xxo} + \varphi_{21}\rho_{yyo}\right),\\
M_{yy} &= s_{22}\left(\varepsilon_{yyo} + \varkappa_{12}\varepsilon_{xxo}\right) - i_{22}\left(\rho_{yyo} + \varphi_{12}\rho_{xxo}\right),\\
M_{xy} &= -s_{12}\gamma_{xyo} + 2\, i_{12}\rho_{xyo},\\
M_{yx} &= -s_{21}\gamma_{xyo} + 2\, i_{21}\rho_{yxo}.
\end{aligned}
$$

where - to comply with notation in ref. /1/

$$\varepsilon_{xxo} = \acute{u}, \quad \varepsilon_{yyo} = \dot{v}, \quad \rho_{xxo} = \ddot{w}, \quad \rho_{yyo} = \overset{\circ\circ}{w}, \qquad /1.7/$$

$$\gamma_{xyo} = \mathring{u} + v, \quad \rho_{xyo} = \mathring{w}!$$

On introducing stress function Φ in the solution of the problem, we obtain with its aid the forces from well-known simple relations

$$N_{xx} = \frac{\partial^2 \Phi}{\partial y^2}, \quad N_{yy} = \frac{\partial^2 \Phi}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \qquad /1.8/$$

Expressing by derivatives Φ and w quantities appearing in the equation of compatibility

$$\frac{\partial^2 \varepsilon_{xxo}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yyo}}{\partial x^2} = \frac{\partial^2 \gamma_{xyo}}{\partial x \partial y} \qquad /1.9/$$

and in the equation of equilibrium of moments

$$\frac{\partial^2 M_{xx}}{\partial x^2} - \frac{\partial^2}{\partial x \partial y}(M_{xy} + M_{yx}) + \frac{\partial^2 M_{yy}}{\partial y^2} = -p \qquad /1.10/$$

equations of functions Φ and w after rearrangement assume the following ~~from~~ form

$$a_1 \frac{\partial^4 \Phi}{\partial x^4} + a_2 \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + a_3 \frac{\partial^4 \Phi}{\partial y^4} + b_1 \frac{\partial^4 w}{\partial x^4} + b_2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + b_3 \frac{\partial^4 w}{\partial y^4} = 0$$

$$/1.11/$$

$$b_1 \frac{\partial^4 \Phi}{\partial x^4} + b_2 \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + b_3 \frac{\partial^4 \Phi}{\partial y^4} - c_1 \frac{\partial^4 w}{\partial x^4} - c_2 \frac{\partial^4 w}{\partial x^2 \partial y^2} - c_3 \frac{\partial^4 w}{\partial y^4} = -p,$$

where

$$a_1 = \frac{1}{(1-\nu_{12}\nu_{21})f_{22}}$$

$$a_2 = \frac{2}{f_{12}+f_{21}} - \frac{2\nu_{12}}{(1-\nu_{12}\nu_{21})f_{11}} \qquad /1.12/$$

$$a_3 = \frac{1}{(1-\nu_{12}\nu_{21})f_{11}}$$

$$b_1 = \frac{s_{11}(\varkappa_{21}-\nu_{21})}{f_{22}(1-\nu_{12}\nu_{21})},$$

$$b_2 = \frac{s_{11}(1-\nu_{12}\varkappa_{21})}{f_{11}(1-\nu_{12}\nu_{21})} + \frac{s_{22}(1-\nu_{21}\varkappa_{21})}{f_{22}(1-\nu_{12}\nu_{21})} - 2\,\frac{s_{12}+s_{21}}{f_{12}+f_{21}},$$

$$b_3 = \frac{s_{22}(\varkappa_{12}-\nu_{12})}{f_{11}(1-\nu_{12}\nu_{21})}, \qquad /1.13/$$

$$d_1 = i_{11} - \frac{s_{11}^2(1-\nu_{12}\varkappa_{21})}{f_{11}(1-\nu_{12}\nu_{21})} - \frac{s_{11}s_{22}\varkappa_{12}(\varkappa_{21}-\nu_{21})}{f_{22}(1-\nu_{12}\nu_{21})},$$

$$d_2 = 2(i_{11}f_{21}+i_{12}+i_{21}) - \frac{s_{11}^2\varkappa_{21}(1-\nu_{12}\varkappa_{21})}{f_{11}(1-\nu_{12}\nu_{21})} - \frac{s_{22}^2\varkappa_{12}(1-\nu_{21}\varkappa_{12})}{f_{22}(1-\nu_{12}\nu_{21})}$$

$$- \frac{s_{11}s_{22}(\varkappa_{21}-\nu_{21})}{f_{22}(1-\nu_{12}\nu_{21})} - \frac{s_{11}s_{22}(\varkappa_{12}-\nu_{12})}{f_{11}(1-\nu_{12}\nu_{21})} - 2\,\frac{(s_{12}+s_{21})^2}{f_{12}+f_{21}},$$

$$d_3 = i_{22} - \frac{s_{22}^2(1-\nu_{21}\varkappa_{21})}{f_{22}(1-\nu_{12}\nu_{21})} - \frac{s_{11}s_{22}\varkappa_{21}(\varkappa_{12}-\nu_{12})}{f_{11}(1-\nu_{12}\nu_{21})}$$

$$/1.14/$$

Introducing

$$\Phi = -\left(b_1 \frac{\partial^4}{\partial x^4} + b_2 \frac{\partial^4}{\partial x^2 \partial y^2} + b_3 \frac{\partial^4}{\partial y^4}\right) \mathcal{U} \equiv - D_\Phi \mathcal{U}, \qquad /1.15/$$

$$w = \left(a_1 \frac{\partial^4}{\partial x^4} + a_2 \frac{\partial^4}{\partial x^2 \partial y^2} + a_3 \frac{\partial^4}{\partial y^4}\right) \mathcal{U} \equiv D_w \mathcal{U}$$

eqs. /1.11/ can be written so as to yield solution of one function $\mathcal{U}$.

This identically fulfils the first of eqs. /1.11/, and substituting /1.15/ in the second equation of /1.11/ we obtain a conditional equation for $\mathcal{U}$.

$$A_1 \frac{\partial^8 \mathcal{U}}{\partial x^8} + A_2 \frac{\partial^8 \mathcal{U}}{\partial x^6 \partial y^2} + A_3 \frac{\partial^8 \mathcal{U}}{\partial x^4 \partial y^4} + A_4 \frac{\partial^8 \mathcal{U}}{\partial x^2 \partial y^6} + A_5 \frac{\partial^8 \mathcal{U}}{\partial y^8} = p \qquad /1.16/$$

Coefficients A_i denote

$$\begin{aligned} A_1 &= a_1 d_1 + b_1^2 , \\ A_2 &= a_1 d_2 + a_2 d_1 + 2 b_1 b_2 , \\ A_3 &= a_1 d_3 + a_2 d_2 + a_3 d_1 + b_2^2 + 2 b_1 b_3 , \\ A_4 &= a_2 d_3 + a_3 d_2 + 2 b_2 b_3 , \\ A_5 &= a_3 d_3 + b_3^2 . \end{aligned} \qquad /1.17/$$

All forces, moments and displacements may be expressed by function $\mathcal{U}$; hence problems with arbitrary boundary conditions can be solved in this form.

For the components of strengths in accordance with /1.8/ and using /1.15/ we obtain

$$
\begin{aligned}
N_{xx} &= -b_1 \frac{\partial^6 U}{\partial x^4 \partial y^2} - b_2 \frac{\partial^6 U}{\partial x^2 \partial y^4} - b_3 \frac{\partial^6 U}{\partial y^6} \\
N_{yy} &= -b_1 \frac{\partial^6 U}{\partial x^6} - b_2 \frac{\partial^6 U}{\partial x^4 \partial y^2} - b_3 \frac{\partial^6 U}{\partial x^2 \partial y^4} \\
N_{xy} &= b_1 \frac{\partial^6 U}{\partial x^5 \partial y} + b_2 \frac{\partial^6 U}{\partial x^3 \partial y^3} + b_3 \frac{\partial^6 U}{\partial x \partial y^5}
\end{aligned}
\qquad /1.18/
$$

Expressing from the first 2 expressions /1.6/ by help of $\varepsilon_{xx,0} = \frac{\partial u_0}{\partial x}$ and $\varepsilon_{yy0} = \frac{\partial v_0}{\partial y}$ we obtain

$$
\begin{aligned}
\frac{\partial u_0}{\partial x} &= \frac{1}{(1-\nu_{12}\nu_{21})f_{11}} \left\{ \frac{s_{11}}{f_{22}} \frac{\partial^6 U}{\partial x^6} - 2\,\frac{(s_{12}+s_{21}+s_{11}\varkappa_{21})-s_{11}}{f_{12}+f_{21}} \frac{\partial^6 U}{\partial x^4 \partial y^2} - \right. \\
&\qquad \left. - \left[\frac{s_{22}}{f_{22}} - 2\,\frac{s_{12}+s_{21}+s_{22}(\varkappa_{12}-\nu_{12})}{f_{12}+f_{21}} \right] \frac{\partial^6 U}{\partial x^2 \partial y^4} \right\}, \qquad /1.19/ \\
\frac{\partial v_0}{\partial y} &= \frac{1}{(1-\nu_{12}\nu_{21})f_{22}} \left\{ \frac{s_{22}}{f_{11}} \frac{\partial^6 U}{\partial y^6} - 2\,\frac{(s_{12}+s_{21}+s_{22}\varkappa_{12})\nu_{21}-s_{22}}{f_{12}+f_{21}} \frac{\partial^6 U}{\partial x^2 \partial y^4} - \right. \\
&\qquad \left. - \left[\frac{s_{11}}{f_{11}} - 2\,\frac{s_{12}+s_{21}+s_{11}(\varkappa_{21}-\nu_{21})}{f_{12}+f_{21}} \right] \frac{\partial^6 U}{\partial x^4 \partial y^2} \right\}.
\end{aligned}
$$

and from this by the help of /1.7/ we obtain the integral of the component of displacement.

The expression of moments is more suitable in the form with the function Φ and w. It is better to substitute for the numeric values of coefficients of solved tasks according to /1.15/. For the moments runs the equation

/1.20/

$$M_{xx} = \frac{s_{11}(1-\nu_{12}\varkappa_{21})}{f_{11}(1-\nu_{12}\nu_{21})}\frac{\partial^2\Phi}{\partial y^2} + \frac{s_{11}(\varkappa_{21}-\nu_{21})}{f_{22}(1-\nu_{12}\nu_{21})}\frac{\partial^2\Phi}{\partial x^2} -$$

$$-\left[i_{11} - \frac{s_{11}^2(1-\nu_{12}\varkappa_{21})}{f_{11}(1-\nu_{12}\nu_{21})} - \frac{s_{11}s_{22}\varkappa_{12}(\varkappa_{21}-\nu_{21})}{f_{22}(1-\nu_{12}\nu_{21})}\right]\frac{\partial^2 w}{\partial x^2} -$$

$$-\left[i_{11}\varphi_{21} - \frac{s_{11}^2\varkappa_{21}(1-\nu_{12}\varkappa_{21})}{f_{11}(1-\nu_{12}\nu_{21})} - \frac{s_{11}s_{22}(\varkappa_{21}-\nu_{21})}{f_{22}(1-\nu_{12}\nu_{21})}\right.\quad \frac{\partial^2\Phi}{\partial y^2}\ /$$

$$M_{yy} = \frac{s_{22}(1-\nu_{21}\varkappa_{12})}{f_{22}(1-\nu_{12}\nu_{21})}\frac{\partial^2\Phi}{\partial x^2} + \frac{s_{22}(\varkappa_{12}-\nu_{12})}{f_{11}(1-\nu_{12}\nu_{21})}\frac{\partial^2\Phi}{\partial y^2} -$$

$$-\left[i_{22} - \frac{s_{22}^2(1-\nu_{21}\varkappa_{12})}{f_{22}(1-\nu_{12}\nu_{21})} - \frac{s_{11}s_{22}\varkappa_{21}(\varkappa_{12}-\nu_{12})}{f_{11}(1-\nu_{12}\nu_{21})}\right]\frac{\partial^2 w}{\partial y^2} -$$

/1.20/

$$-\left[i_{22}\varphi_{12} - \frac{s_{22}^2\varkappa_{12}(1-\nu_{21}\varkappa_{12})}{f_{22}(1-\nu_{12}\nu_{21})} - \frac{s_{11}s_{22}(\varkappa_{12}-\nu_{12})}{f_{11}(1-\nu_{12}\nu_{21})}\right]\frac{\partial^2 w}{\partial x^2}.$$

$$M_{xy} = 2\frac{s_{12}}{f_{12}+f_{21}}\frac{\partial^2\Phi}{\partial x\partial y} + 2\left(i_{12} - s_{12}\frac{s_{12}+s_{21}}{f_{12}+f_{21}}\right)\frac{\partial^2 w}{\partial x\partial y},$$

$$M_{yx} = 2\frac{s_{21}}{f_{12}+f_{21}}\frac{\partial^2\Phi}{\partial x\partial y} + 2\left(i_{21} - s_{21}\frac{s_{12}+s_{21}}{f_{12}+f_{21}}\right)\frac{\partial^2 w}{\partial x\partial y}.$$

Solution of eq. /1.16/ for an orthotropic plate of the bridge type according to Fig. 1, freely supported along edges paralel to y with boundary conditions

Fig. 1.

$$[(x=0,\ x=a)\,y] \qquad w=0,\ M_{xx}=0,\ N_{xx}=0,\ T_{xy}=0\ / \qquad /1.21/$$

$$[x\,(y=0,\ y=b)] \qquad M_{yy}=0,\ N_{yy}=0,\ A_{y}=0,\ T_{yz}=0\ .$$

Both the boundary conditions and the equation are satisfied by solutin

$$U_{(x,y)} = U_{o(x,y)} + U_{1(x,y)} \qquad /1.22/$$

where $U_{o(x,y)}$ is a general solution of homogeneous equation and $U_{1(x,y)}$ a particular solution of non--homogeneous equation /1.16/ if the solutions are considered in the form

$$U_{o(x,y)} = \sum_{m=1}^{\infty} \sin \lambda x \sum_{i=1}^{i=4} \left(B_i \, ch\,\lambda k_i y + C_i \, sh\,\lambda k_i y \right), \qquad /1.23/$$

$$U_{1(x,y)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} \sin \lambda x \sin \nu y\ .$$

Here $\lambda = \frac{m\pi}{a}$, $\nu = \frac{n\pi}{b}$, m, n are positive interers.

2. Determining the physical and geometric parameters /1.1/

In view of greatly different geometric configuration of transverse cross sections, no general method of determining the parameters of rigidity can be established. Difficulties are never encountered when determining rigidities $f_{11}, f_{22}, s_{11}, s_{22}, i_{11}, i_{22}$ which - using notation of ref. /1/ for the case in accordance with Fig. 1 - have the following values

$$f_{11} = \frac{Eh}{1-\nu^2} + EA_x \quad , \quad f_{22} = \frac{Eh}{1-\nu^2} + EA_y \qquad /2.1/$$

$$s_{11} = ES_x , \qquad s_{22} = ES_y ,$$

$$i_{11} = I + I_x , \qquad i_{22} = I + I_y .$$

In the majority of cases, coefficients of unit values of transverse rigidity pertain to the plate only; if however, the stiffeners of the plate are closely spaced, the effect of transverse rigidity of the stiffeners may also be evidenced. For the case in accordance with Fig. 1 of ref. /1/

$$\nu_{12} = \nu_{21} = \nu , \quad \varkappa_{12} = \varkappa_{21} = 0 , \quad \varphi_{12} = \varphi_{21} = \nu . \qquad /2.2/$$

Considerable difficulties are experienced in the determination of the rigidity under shearing stress. In case the stiffeners are quite thin, their shearing rigidity is equal zero and the shearing rigidity of the orthotropic plate is determined only from the rigidity of the plate.

$$f_{12} = f_{21} = \frac{Eh}{2(1+\nu)} \qquad /2.3/$$

In case the stiffeners are not thin and are closely spaced, their shearing rigidity must also be considered; the authors respect it by value $\frac{EA}{3(1-\nu)}$.

If the stiffeners are closely spaces in both directions, the shearing rigidity is further increased by the bending of the stiffeners in the plate plane /Fig. 2/.

$$f_{21} = \frac{12 E J_{z,x}}{l_{ox}^2} \,,$$

$$f_{12} = \frac{12 E J_{z,y}}{l_{oy}^2}$$

Fig. 2.

From the same assumptions it follows for the rigidities of the orthotropic plate in accordance with Fig. 1 of ref. /1/ that

$$S_{12} = S_{21} = 0 \qquad /2.4/$$

and, furthermore, that

$$i_{12} = i_{21} = \frac{E h^3}{12(1+\mu)} \qquad /2.5/$$

The advantage of the foregoing formulation lies in that it is not necessary to set up equations for each particalar case and that problems may be solved by a single function for any arbitrary boundary conditions.

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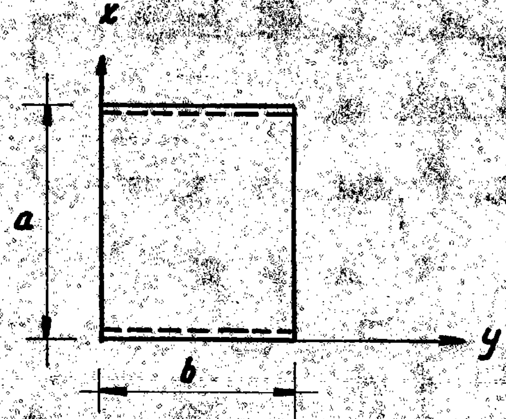
x
a
b
y

Fig. 1.

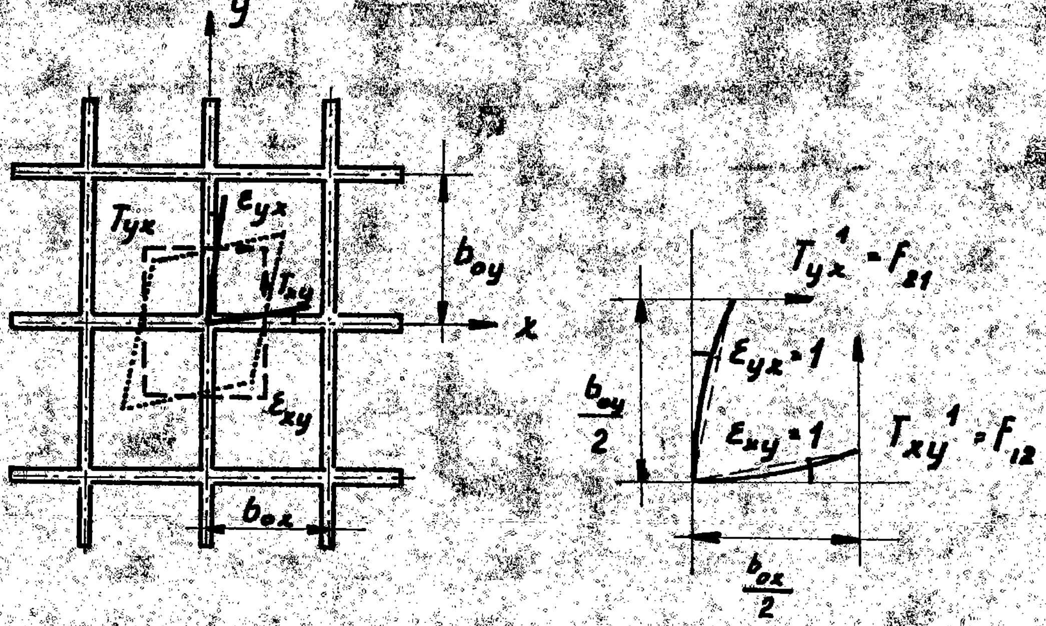
y
x
T_{yx}
ε_{yx}
T_{xy}
ε_{xy}
b_{oy}
b_{ox}
$T_{yx}^{1} = f_{21}$
$\varepsilon_{yx} = 1$
$\varepsilon_{xy} = 1$
$T_{xy}^{1} = f_{12}$
$\frac{b_{oy}}{2}$
$\frac{b_{ox}}{2}$

Fig. 2

Summary

All up to date known analytic methods for investigating systems built up of plates stiffened by beams rigidly connected to the plates process two common deficiency - the analysis is not sufficiently general while the numerical work involves many difficulties. In this paper a method of analysis is discussed which otherwise complies with the usual theory of orthotropic plates, but also the sheat stiffness of the slab is taken into account.

The present method of analysis is based on the idea that the integrals of internal forces must be equivalent for both the actual and the substitute, idealized plates. The substitute plate idealizes the actual orthotropic plate, and equivalency must be obtained for all the constants of the cross - sections, whether actual, or idealized. The differencial equation that governs the problem has been derived with due respect to and by logical combining of both - the plate and the wall - effects.

A practical example is given, where the solution for boundary conditions is shown for a bridge slab stiddened by beams.

The proposed method of analysis is at present being cheched by experimental tests using plexiglas models, so that reliable data will be available for this type of structures, the test being caried out for a wide range of both the dimension and of the loads.